Day 1

Thinking Dynamically

A (discrete-time) dynamical system is a function from a set X to itself

$$f : X \to X$$

function

space (things that get plugged in)

same space (thing that come out)

Example: X = all real numbers aka the number line aka $\mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ is $f(x) = \frac{1}{2}x$

Orbits: the evolution of the system, for the point $x_0 = 6$ is

$x_0 = 6$, $x_1 = f(x_0) = \frac{1}{2}(6) = 3$, $x_2 = f(x_1) = \frac{1}{2}(3) = \frac{3}{2}$

$x_4 = f(x_3) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$

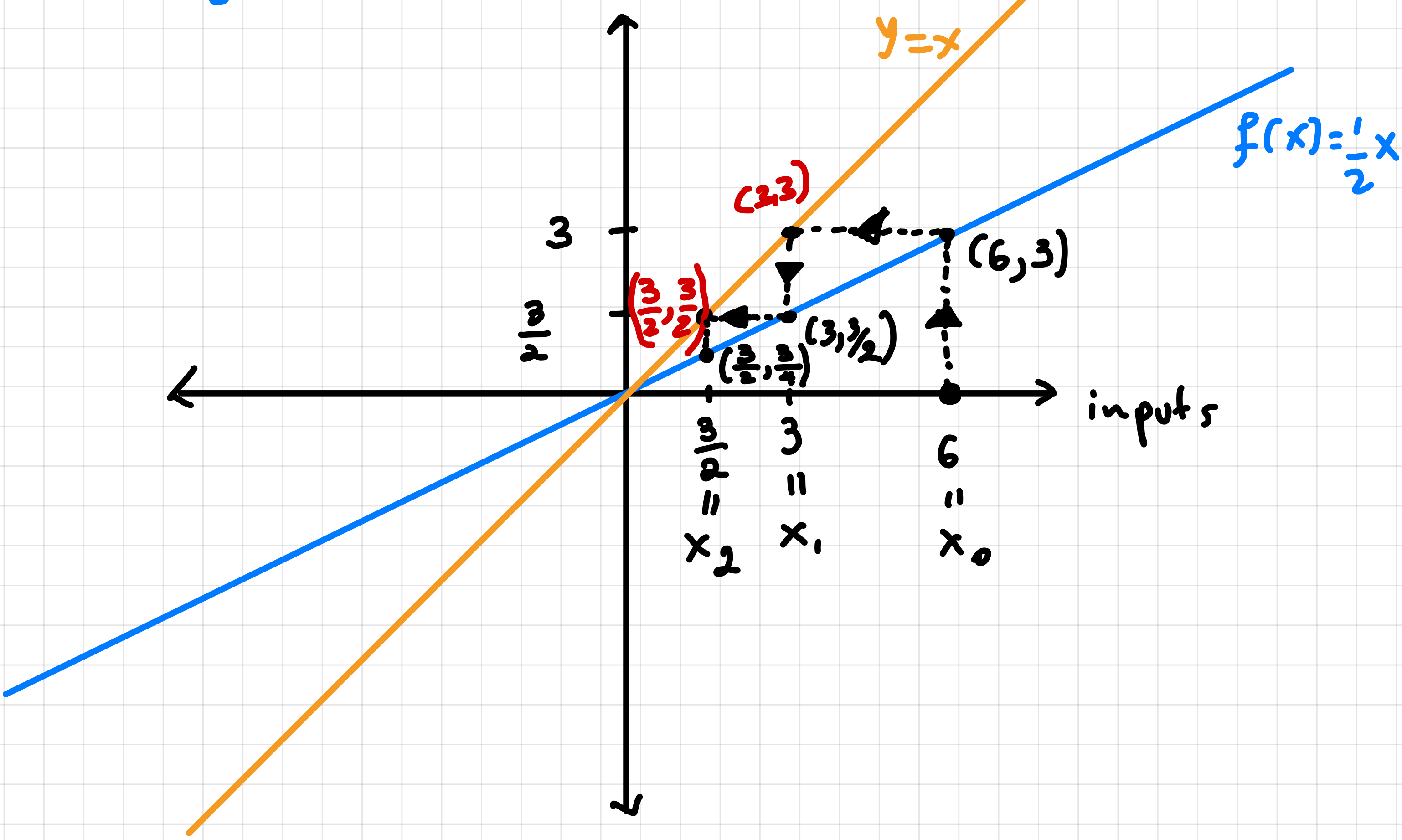
These orbits are related to the geometric sequence

Notation:
- $f^k(x)$ = k^{th}-step in the orbit of x.

  Warning: $f^k(x) \neq (f(x))^k$

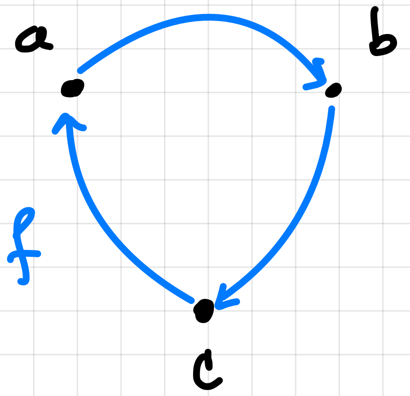
- The (forward) orbit of a point x is the set $\mathcal{O}(x) = \{x, f(x), f^2(x), \dots\}$

Drawing iterates



Example: X = a finite set $\{a, b, c\}$

$f: X \to X$ is defined by the arrows in the following grap



$f(a) = b$

$f(b) = c$

$f(c) = a$

Orbits

$a, \quad f(a) = b, \quad f^2(a) = f(f(a)) = f(b) = c, \quad f^3(a) = a$

$b, \quad f(b) = c, \quad f^2(c) = a, \quad f^3(b) = b$

$c, \quad f(c) = a \quad f^2(c) = b, \quad f^3(c) = c$

in this case $f^3 = id$ ← identity transformation

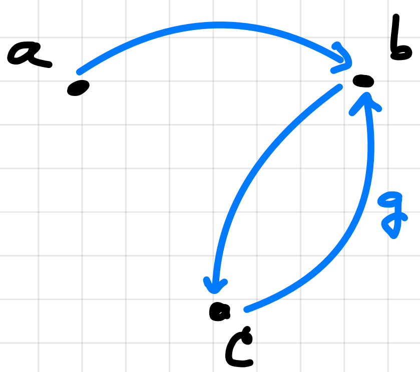
Definition A point p is called *periodic* if $f^n(p) = p$ for some positive integer n. The smallest such integer is called the period.

Example for $f(x) = \frac{1}{2}x$ has only one periodic point, $x = 0$.

Example

$X = \{a, b, c\}$

$g: X \to X$



b & c are periodic with period 2.

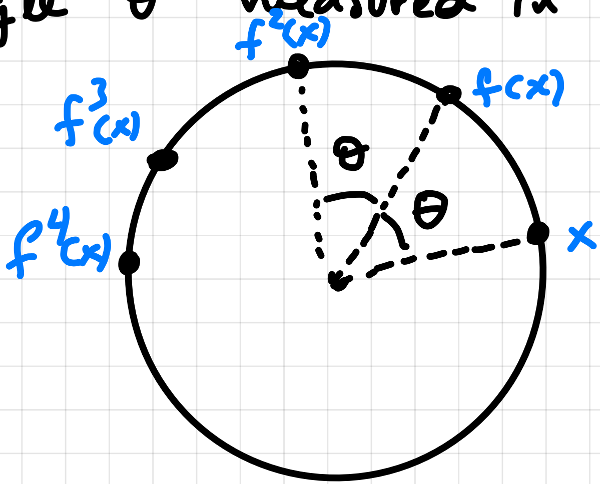
a is preperiodic: $f^2(a) = f^4(a)$

Definition a point x is *pre-periodic* if there are two integers $m \neq n$ with

$$f^m(x) = f^n(x)$$

Example X = unit circle $= \{(x,y) \in \mathbb{R}^2 ;\ x^2 + y^2 = 1\}$

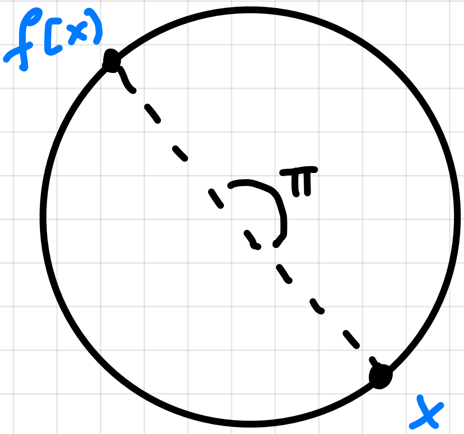
Pick an angle θ measured in radians



$f: X \to X$ is the counter-clockwise rotation of x at angle θ.

Also called circle rotation with angle θ.

If $\theta = \pi$, then



Example $X =$ The space of finite subsets of rational numbers

$p \in X$ then p can be $p = \{\frac{1}{3}, \frac{2}{5}, \frac{4}{7}\}$

f is defined by (for example):

$$f\left(\left\{\frac{1}{3}, \frac{2}{5}, \frac{4}{7}\right\}\right) = \left\{\frac{1}{3}, \frac{1+2}{3+5}, \frac{2}{5}, \frac{2+4}{5+7}, \frac{4}{7}\right\}$$

$$= \left\{\frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7}\right\}$$

input: is a finite set in increasing order

output: the initial set and the medians of two consecutive initial elements.

Some interesting thoughts:

- Start with any pair $\{a,b\}$. Do you eventually see all rational numbers between a and b?
- what happens to the gaps between succesive elements?
- what is the denominator size? How quickly do they grow?

Periods & Periodic Points

Day 2

Recall p is periodic for $f: X \to X$ if

$$f^k(p) = p \quad \text{for some } k$$

The minimal & positive such k is called the period of p.

Consider a periodic point p.

Let $A(p)$ denote the set of times such that

time

$$k \in A(p) \quad \text{if and only if} \quad f^k(p) = p$$

time

Observations

- $A(p)$ is a set of natural numbers
- The smallest time is the period of p.

Claim: $A(p) = \{\overset{\text{period of } p}{n}, 2n, 3n, 4n, \ldots\}$

where n is the period of p

we want to prove the claim using set theory:

<u>Subsets</u> Two sets A, B.

$A \subset B$ if every element of A belongs to B

Also, $A = B$ if $A \subset B$ & $B \subset A$.

<u>Showing</u>: $\{n, 2n, 3n, \ldots\} \subset A(p)$

Since n is is the smallest element of $A(p)$

$$f^n(p) = p$$

Now, for every k,

$$f^{kn}(p) = \underbrace{f(f(\ldots(f(p})_{kn \text{ times } f}) \ldots))$$

$$= \underbrace{f^n(f^n(\cdots f^n}_{k \text{ times } f^n}(p) \cdots))$$

$$= \underbrace{f^n(f^n(\ldots}_{k-1 \text{ times } f^n}(p) \cdots)) \quad \text{Since } f^n(p) = p$$

k times:

$$= p$$

Showing: $A(p) \subset \{n, 2n, \dots\}$

We will use a technique called contradiction

Suppose that there exists $m \in A(p)$ which is not in $\{n, 2n, \dots\}$. This means that m is not divisible by n

Let $\ell = \lfloor \frac{m}{n} \rfloor$

and $q = m - \ell n$

($\lfloor \cdot \rfloor$ is the floor.)

for example $\lfloor 3.5 \rfloor = 3$

$\lfloor 7 \rfloor = 7$, $\lfloor 8.2 \rfloor = 8$

Exercise $0 < m - \ell n = q < n$

Continuing with the proof,

$$f^q(p) = f^q(f^{\ell n}(p)) = f^{q+\ell n}(p) = f^{m-\ell n+\ell n}(p) = f^m(p) = p$$

So $f^q(p) = p$ for some number smaller than n, i.e. q.

Since $0 < q < n$, this contradicts n being the smallest element of $A(p)$.

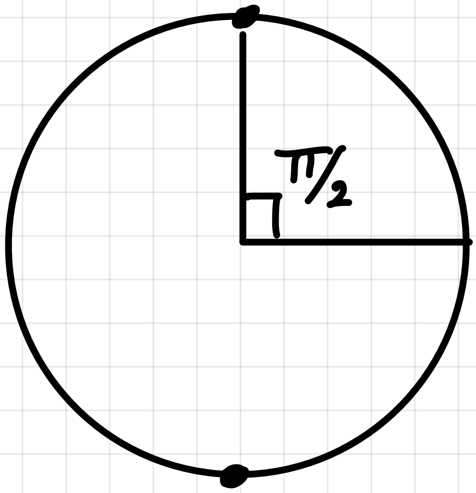
And so, we must have $A(p) \subset \{n, 2n, \dots\}$

Exercise: If $f^k(p) = p$, then the period of p divides k. (i.e., k is divisible by the period of p).

Understand Spaces

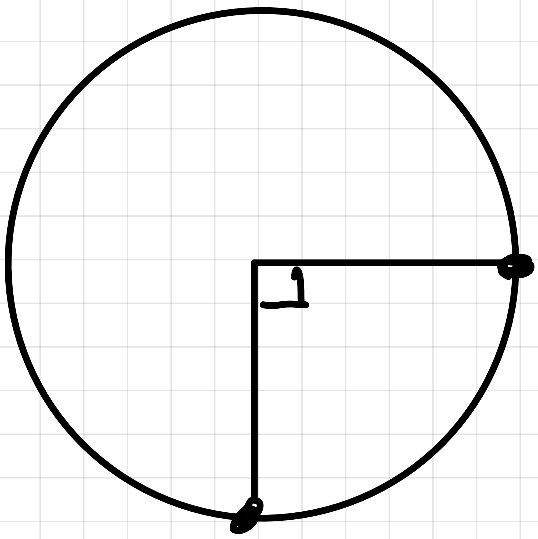
The Circle

"$\frac{\pi}{2}$"



"$-\frac{\pi}{2}$" or "$\frac{3\pi}{2}$"

A point on the circle is the *set* of all angles realizing that point.



$\{\ldots, -2\pi, 0, 2\pi, 4\pi, \ldots\}$

$=$

$[0]$ equivalence class of 0.

$\{\ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots\}$

$=$

$[-\frac{\pi}{2}]$ equivalence class of $-\frac{\pi}{2}$

<u>Notation</u> $\mathbb{R}/_{2\pi\mathbb{Z}}$ denotes the set of equivalence classes

integers